Encouraging Meaningful Quantitative Problem Solving

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Overview: The first part of this presentation, encourages readers to reflect upon their own problem-solving strategies by answering the five questions in Problem Set 1. (Of course this and the other activities are optional.) A set of possible solutions that represents a range of traditional approaches to the five questions is then presented, along with some potential sources of difficulty that they may pose for novice problem solvers. Next, in Problem Set 2, a second set of problems is presented and discussed, which anticipates an approach that might be used in guiding students to become better quantitative problem solvers. The approach, based on the ideas of Arons (ref 1), is then introduced, and Problem Set 1 is revisited. The final part of the presentation describes insights gained by experimenting with this approach during the past ten years, and some of the resulting modifications that have been made in the freshman chemistry curriculum (ref 2).

Problem Set 1: Show how you would solve the following problems.

1. A car traveled 270 miles in 5 hours. How many hours did it take the car to go 150 miles? (Assume it traveled at constant velocity.)

2. Convert 3000 meters into kilometers.

3. The density of a particular material is 2.5 grams / cubic cm. What is the volume of 500 g of this material?

4. Consider the balanced chemical equation for the reaction of sulfuric acid with sodium hydroxide:

H_2SO_4 + 2 NaOH ---> Na_2SO_4 + $2H_2O$

Find the number of moles of H_2SO_4 that react with 5 moles of NaOH.

5. Let x = number of meters and y = number of kilometers. An equation using the symbols x, y, and 1000, which expresses the relationship between number of meters and number of kilometers is
a) 1000x = y b) 1000y = x c) x + y = 1000 d) xy =1000

Solutions to Problem Set 1

Although each of these problems can be solved in several ways, here are a set of solutions that represents a range of traditional approaches often suggested to students. Many students bring such approaches with them to their first-year college chemistry courses. You may have used other approaches, and might even object to some of the ones presented here. Different approaches will be considered later in the presentation.

problem 1. Using **ratio and proportion**:
The student is told to set up a proportion, cross multiply, and solve for x:
270 mi 150 mi
5hr x hr 270x= 150(5)
x = 2.78 hr

Using dimensional (unit) analysis, also known as the **factor-label method**:
The student is told to set up an expression in which the units cancel to produce the desired unit. In this case the unit mi cancels when the terms on the left side of the equal sign are multiplied:

5 hr x 150
mi= 2.78 hr
270 mi

problem 2. Using **unitary conversion** (or unit-factor method):
The student is told that since 1 km = 1000 m, 1 km / 1000 m is equal to 1, and so multiplying 3000 m by this factor does not change the distance. It merely changes the unit.

3000 m (1 km)
= 3 km therefore,
(1000 m)

3000
m =3 km

A second method would have the student focus on canceling the unit m leaving the unit m, in a fashion similar to that described in the second solution to problem 1 above.

problem 3. Using a **plug and chug** calculation:
The student is told to plug known quantities into a given equation and solve for the unknown quantity:
D = m / v
2.5 g / cc= 500 g / **x**
cc **x** = 200 cc

problem 4. According to the balanced chemical equation, 1mole H_2SO_4 = 2 moles NaOH. (This is the concept of **reaction chemical equivalents**.) Dividing both sides of this equation by 2 moles NaOH,we get:

(1mole H_2SO_4) / (2 moles NaOH) = 1 (as in problem 2)

To find the number of moles of H_2SO_4 that react with 5 moles of NaOH we can write:
(5 moles NaOH)**(1mole H_2SO_4)** = ? moles H_2SO_4
(2 moles NaOH)
and it follows that 5.0 moles of NaOH=2.5 moles H_2SO_4 (as in problem 2)
The ratio & proportion or unit analysis methods, used in problem 1, can also be used here.

problem 5. b is correct. In words, the equation states that 1000 times the **number of** kilometers equals the number of meters.
Answer a is frequently chosen.In this case, x and y are mistaken for labels, instead of variables

While any of these approaches is fine in the hands of an individual who has a clear picture of the physical situation associated with the problem, each of them has the potential to become an exercise in mere symbol manipulation when used by novice problem solvers (refs 1 and 2). In addition, each has unique problems associated with its use. For example, few students can express why it is all right to cross multiply in problem 1. Interpreting ratios as being equal to one (used in problems 2 and 4) is mathematically incorrect in problem 4, and encourages the concept of reaction chemical equivalents, which the chemical education community in this country has abandoned. Dimensional analysis, ratio and proportion, and plug-in type calculations may lead to correct numerical answers without a clear picture of the physical situation associated with a problem, and the inability to write correct equations. Lack of skill in equation writing leads most students to choose the incorrect answer a, in problem 5, which mistakes x and y as labels instead of variables.

Is there any feature, common to all of these problems, that we can draw on in an effort to guide students toward becoming better problem solvers? Before we consider this question, please solve Problem Set 2.

Problem Set 2: How would you solve the following problems?
6. Using only the set of circles below, show how you would represent the operation,
6 2.
O O O O O O
Explain in words what you are doing, without using the word division.

7. Based on the definition of division, draw a picture that justifies the fact that,
2 1 / 2 = 4

8. Consider a group of 40 apples and a group of 5 apples. Evaluate the following ratio and state what your answer means: 40 apples / 5 apples.

9. Consider the ratio, $40 / 5 gallons. Evaluate this ratio and state what your answer means. Now consider the ratio 5 gallons / $40. Evaluate this ratio and state what this answer means.

10. A certain liquid costs $5 per gallon. What does the following expression represent? $40 / ($5 / 1 gallon) When you perform the division, what is the label on the number that you obtain? Explain why. How could you get the same result by a different method? (i. e. do not divide 40 by 5.)

Arons (1) suggests that students struggle with quantitative problem solving because they are unable to articulate the meaning of division and various ratios. With this in mind, let s look at the answers to problems 6-10:

Solutions to Problem Set 2

problem 6. Separate the circles this way: OO / OO / OO
The word quotient comes from Latin and means how many times? In this case, the quotient, three, refers to the fact that we can remove two circles a total of **three times** from six circles. Many individuals draw only one line, dividing the group of six circles in half. In this case two would represent the number of groups, and three would represent the number of circles in each group.

problem 7. When we divide a number by a fraction, the rule is to invert the fraction and multiply the number by it: 2
1/2 = 2 x (2/1) = 4
We can see why this works when we understand the meaning of division. The problem is asking how many times can 1/2 be removed from 2? While most students can produce the answer symbolically by inverting and then multiplying, no student in the past several years of freshman chemistry has been able to give a qualitative description in which two identical objects are cut in half, allowing half an object to be removed from the group a total of four times.

problem 8. A ratio expresses a comparison or relationship between different quantities. We divide the quantities to evaluate the ratio. In this case, 40 apples / 5 apples = 8 Students often incorrectly indicate that the label on 8 is apples. The number 8 indicates that a group of 40 apples is **8 times** larger than a group of 5 apples. To reinforce this idea, a proportion can be helpful:
40 apples = 8 apples
5 apples 1
apple

We can say that for each apple in the denominator group, there are 8 apples in the numerator group. It is also useful to indicate that 8 x 5 apples = 40 apples

problem 9. Dividing 40 by 5, we get 8. This represents the fact that $8 will purchase 1 gallon. Dividing 5 by 40, we get 0.125. This represents the number of gallons that can be purchased for $1. Many college freshmen struggle when they are asked to express the meaning of the number obtained when the quantities in a ratio are divided. Often their response involves repeating the original information, or saying that they do not know. It helps students to see the proportions
$40 = $8 5 gal = 0.125 gal
5 gal 1 gal
$40 $1

problem 10. Here, the idea that division entails repeated subtraction is useful. Each time we give the store owner $5, we get 1 gallon. With $40, we can do this a total of eight times, and receive 8 gallons. To evaluate the set of labels, $ / ($/gal), we do the same thing that is done when dividing by a fraction, we invert the denominator and multiply by the numerator:
$ x (gal / $) = gal. The unit $ cancels and we are left with the unit gallon.
Here is another way to represent the number of gallons that can be purchased for $40:
Invert the ratio $5 / 1 gal: 1 gal / $5 = 0.2 gal / $1 This is the number of gallons that can be purchased for one dollar. For $40 we can get 40 times this amount. Keeping track of the units we write: $40 x (0.2 gal / $1) = 8 gallons. Once again, it is challenging for many first year chemistry students to express these ideas in words.

Revisiting Problem Set 1

Since ratios are fundamental to so many aspects of science, it is important to spend time helping students to learn to articulate their meaning. Unless students can express these in their own words, they are likely to cling to patterns of problem solving based primarily on symbol manipulation. Revisiting problems 1 - 5, we can see that articulating the meaning of ratios can serve as the foundation for expressing solutions to **all** of these problems:

problem 1. Arons (ref 1) recommends asking students to interpret a ratio and its inverse. In this case we have:
(1) 270 mi / 5 hr = 54 mi / 1 hr **or** **(2)** 5 hr / 270 mi = 0.0185 hr / 1 mi

In general, a ratio containing quantities with different labels represents the amount of the numerator for one unit of the denominator.

The second case above gives the time it takes to go **one mile.** If the car travels at constant velocity, it will take 150 times as long to travel 150 miles. Once the student can express this fact, **then** s/he can write a statement that formally keeps track of the units:
0.0185 hr x 150
mi = 2.78 hr
1 mi

Note that merely writing5 hr x 150 mi = 2.78 hr,
270
mi
and cancelling the units does not guarantee that the student can verbalize the reasoning. Setting up the correct proportion, as was done earlier, also does not guarantee that the student can express the reasoning.
A second way to solve this problem would require the student to realize that a 54-mile parcel is traversed each hour, so the number of these parcels in 150 miles corresponds to the number of hours it will take to go 150 miles. Dividing 150 mi by 54 mi we find that there are 2.78 of these parcels in 270 miles, and therefore it will take 2.78 hours to traverse the distance. **After** the student can express this in words, s/he can be guided to the formal statement in which the units work out to be hours: 150 mi / (54 mi / 1 hr) = 2.78 hr. A third approach to this problem involves interpreting the ratio 150 mi / 270 mi, and using this along with the time to go 270 miles, to reason toward the answer.

problem 2.1 km / 1000 m = 0.001 km / 1 m. There is 0.001 kilometer in one meter. In 3000 meters we will have 3000 times this many kilometers:
3000 m x (0.001 km /

1m) = 3 km

Even if one just writes 1 km____ x 3000 m = 3km,
1000
m
as long as 1 km / 1000 m is interpreted as the number of km in 1 m, this reinforces the qualitative meaning. A second way to solve this problem would have the student interpret the expression 3000m / (1000m / 1 km) in a fashion similar to that used in problem 1.

problem 3. Each time we remove a 2.5-g clump of this material, we get 1 cc. From 500 g we find that this can be done 200 times. Keeping track of the units we write:
500 g / (2.5 g / 1cc) = 200
cc

Another way: Invert 2.5 g / 1 cc: 1 cc / 2.5 g = 0.4 cc / 1 g. This is the volume of 1 gram of the material. The volume of 500 g will be 500 times this amount. Keeping track of the units, we write:
(0.4 cc / 1 g) x 500 g =
200 cc.

problem 4. It is true that 2 moles of NaOH corresponds to 1 mole of H_2SO_4 (in older chemical jargon, one would say that 2 moles of NaOH is equivalent to 1 mole of H_2SO_4), but mathematically equating different physical quantities is incorrect. Interpreting a ratio

of two quantities with different labels as the amount of numerator for one unit of the denominator discourages the concept of reaction chemical equivalents. This means we avoid writing
1mole H_2SO_4 / 2 moles NaOH = 1. Instead, interpret the ratio as indicating that 0.5 moles of H_2SO_4 reacts with 1 mole of NaOH. 5 moles of NaOH will therefore require 5 times this much H_2SO_4. Keeping track of the units we write:

(5 moles NaOH)**(0.5mole H_2SO_4)** = ____________ 2.5 moles H_2SO_4
(1 mole NaOH)

Can you use the ratio, 2 moles NaOH / 1mole H_2SO_4 to solve the problem a different way?

problem 5. Interpret the ratio 1000m / 1km as the number of meters in 1 km, and not as being equal to 1. The value of the ratio of x / y is fixed at 1000 / 1 We write
x / y = 1000 / 1, which can be rearranged to give 1000y = x (which is choice b of the original selections on pg 1 for this problem).

Note that correspondence equations like 1 mol H_2SO_4 = 2 mol NaOH, mentioned in problem 4, often lead students to write incorrect equations.Thus, if x = number of moles of H_2SO_4, and y = number of moles of NaOH, a correct equation is y = 2x, not x = 2y. When students are encouraged to ~~create an~~d interpret ratios, this sets ~~the stage f~~or the correct setup and interpretation of equations. Thus, in this case y / x = 2, and it more easily follows that y = 2x.

Note that there is more than one way to solve each of these problems using ratios. The key is that when the *student* can verbalize the meaning of the ratio, qualitative understanding of the physical situation is enhanced, and the chances of developing misconceptions about the material are reduced. This is an important first step in becoming a successful problem solver.

An approach in which ratios are created and explicitly interpreted offers advantages over the other methods (ref 2). The method avoids the tedium of ratio and proportion when a problem has more than one step. It is also less mechanical than the dimensional analysis method, since it focuses on the relationship between physical quantities, not the relationship between units. The method of unitary conversion, while harmless when used in converting units (problem 2, page 2 above), encourages students to embrace the abandoned idea of chemical equivalents (problem 4, page 2) and other incorrect correspondence equations (e. g. writing $2 = 1 gallon, when milk costs $2 per gallon). The student has to know when to interpret a ratio as being equal to one, and when not. For concrete reasoners this adds a complicating factor to an already difficult task. This suggests that ratios like 1 km / 1000 m in problem 2 should be interpreted as 0.001 km / 1 m (see solution on page 5 above), and that setting them equal to 1 should be avoided. Finally, research indicates that the difficulty students have with setting up and interpreting equations stems from a difficulty with translating words into equations and vice versa, rather than a difficulty with simple algebraic manipulations. When students are encouraged to create and interpret ratios, it is more likely that they will be able to interpret and write equations correctly.

Commentary on Using this Approach over the Last Ten Years

An active learning environment in which instructors can get feedback on student difficulties is important. Students engage in paired problem solving in class, help sessions, and in the laboratory. Socratic lines of questioning are used by instructors to drive students toward constructing their own understanding and approaches. Starting with familiar quantities is important (e.g. dollars, gallons, miles, hours), since unfamiliarity with scientific vocabulary complicates the situation. A multiple choice format is used in testing, since the class size is always greater than 100 students. As a result, qualitative descriptions of phenomena must be explicitly connected with the quantitative symbolic representations. Correct setups are stressed in problems that are assigned as part of laboratory activities.

Students often come to their college freshman chemistry courses with specific approaches to quantitative problem solving, usually ratio & proportion or dimensional analysis. Although the presentation we give stresses ratio interpretation, these other methods are tolerated. Students who cling to these other approaches are shown the logical connections to ratio interpretation. The instructor constantly must be on guard against student use of blind symbol manipulation, while also showing students that at times they can let go of physical pictures.

The number of topics has been reduced and their sequence has been modified, to allow more time for students to develop the ability to articulate the meaning of ratios and apply this skill to quantitative problem solving. For example, the mole concept has been delayed so that students can develop practice in quantitative problem solving first using familiar quantities like dollars and gallons, and then using concepts like density and % composition by mass. Reduced coverage includes more limited presentations of descriptive chemistry, colligative properties, and solution equilibria, and eliminating topics like quantum numbers and an introduction to organic chemistry.

References

1. Arons, A.; *A Guide to Introductory Physics Teaching*, Chapter 1, Wiley: New York, 1990.

2. Garafalo, F. et. al., *Journal of Chemical Education* , Vol.77, Sept. 2000, p1166-1173 (and references therein).